The Triple Product

The **triple product** is not a "new" operation, as it is simply a combination of the **dot** and **cross** products.

The triple product of vectors **A**, **B**, and **C** is **denoted** as:

$$\boxed{\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}}$$

Q: *Yikes! Does this mean:*

$$(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C}$$

or

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$$

A: The answer is **easy** ! Only one of these two interpretations makes sense:

In the **first** case, $\mathbf{A} \cdot \mathbf{B}$ is a scalar value, say $d = \mathbf{A} \cdot \mathbf{B}$. Therefore we can write the first equation as:

$$(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C} = d \times \mathbf{C}$$

But, this makes no sense! The cross product of a **scalar** and a vector has **no meaning**.

In the **second** interpretation, the cross product $\mathbf{B} \times \mathbf{C}$ is a **vector**, say $\mathbf{B} \times \mathbf{C} = \mathbf{D}$. Therefore, we can write the second equation as:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{A} \cdot \mathbf{D}$$

Not only does this make sense, but the result is a **scalar**!

The triple product $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$ results in a **scalar value**.

The Cyclic Property

It can be shown that the triple product of vectors **A, B,** and **C** can be evaluated in three ways:

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A}$$

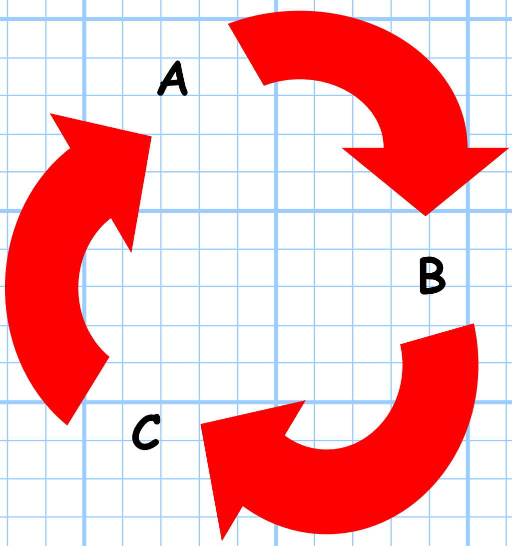
But, it is important to note that this does **not** mean that order is unimportant! For example:

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} \neq \mathbf{A}\cdot\mathbf{C}\times\mathbf{B}$$

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} \neq \mathbf{C}\cdot\mathbf{B}\times\mathbf{A}$$

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} \neq \mathbf{B}\cdot\mathbf{A}\times\mathbf{C}$$

The **cyclical** rule means that the triple product is **invariant** to **shifts** (i.e., rotations) in the order of the vectors.



There are six ways to arrange three vectors. Therefore, we can group the triple product of three vectors into two groups of two:

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} &= \mathbf{C}\cdot\mathbf{A}\times\mathbf{B} = \mathbf{B}\cdot\mathbf{C}\times\mathbf{A}\\ \mathbf{B}\cdot\mathbf{A}\times\mathbf{C} &= \mathbf{C}\cdot\mathbf{B}\times\mathbf{A} = \mathbf{A}\cdot\mathbf{C}\times\mathbf{B}\end{aligned}$$

but,

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = -\left(\mathbf{B}\cdot\mathbf{A}\times\mathbf{C}\right)$$